Metal Forming – BSc 2026/27-1

Calculation methods

Introduction

A full solution would need:

Establish a precise physical model for the given forming task and apply a precise mathematical solution.

This leads to a partial differential equation system.

Generally, such equation systems can't be solved.

Therefore, simplifications are applied to the physical model or the mathematical solution, or even to both.

The simplified solutions are made for the two border strain cases: axisymmetric and plain strain.

Introduction

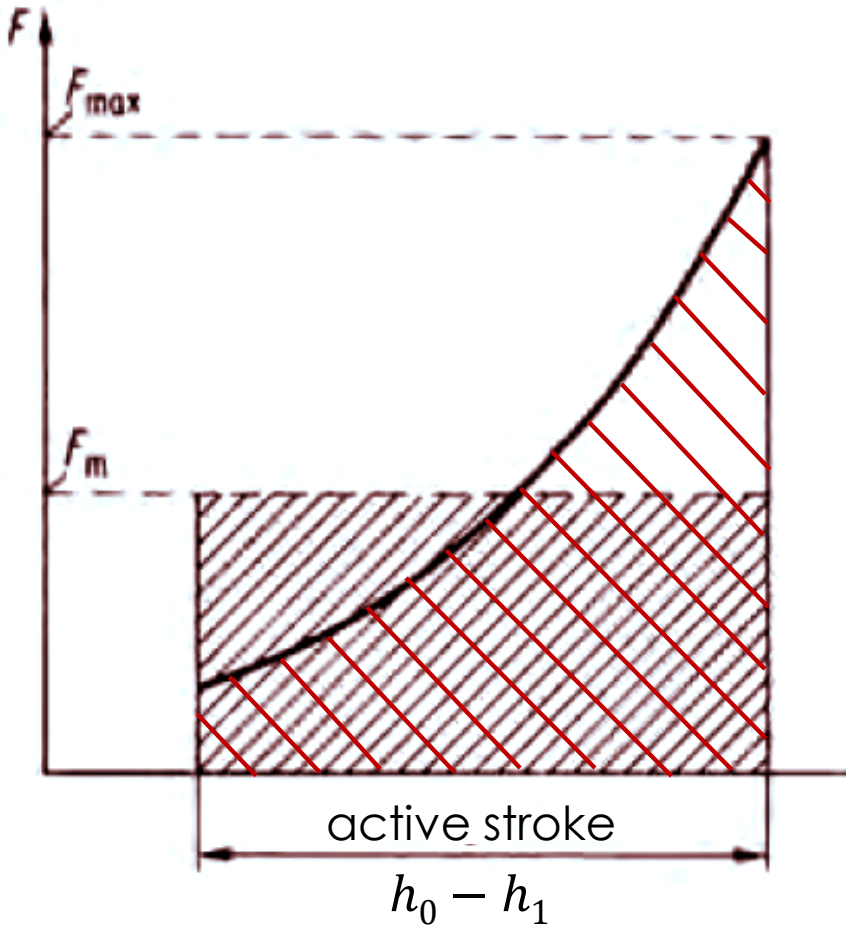
Fast estimation for upsetting:

$$W = \frac{V \cdot \sigma_{fm} \cdot \varphi_{eq}}{\eta_F}$$

or

$$W = F(h_0 - h_1) \cdot x$$

$$x = \frac{F_m}{F_{max}} \qquad x \cong 0.6$$



- W - upsetting work
- V - volume involved in deformation
- σ_{fm} - mean flow stress
- φ_{eq} - equivalent strain
- η_F - deformation efficiency (0.6 – 0.9)
- h_0 - stock height
- x - process factor
- F_m - mean force
- F_{max} - maximum force

Border cases

Simplified solutions (in principal coordinate system):

Axial symmetrical case

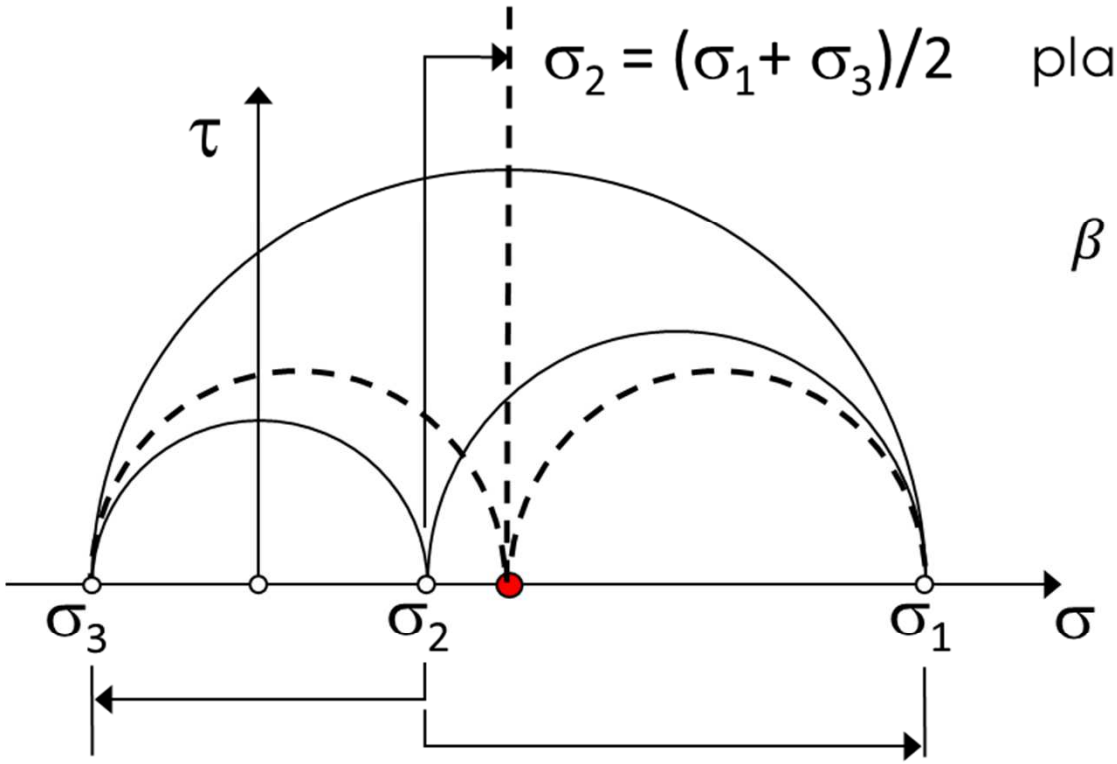
The strains and also the stresses, which are perpendicular to the axis, are identical.

Plain strain

In the second main direction, the strain is zero. To ensure this, the second main stress can not be zero.

Border cases

General flow criterion: $\sigma_1 - \sigma_3 = \beta\, \sigma_f$



Main methods

Solution technics

- Equilibrium Approach
- Energy Approach
- Others …

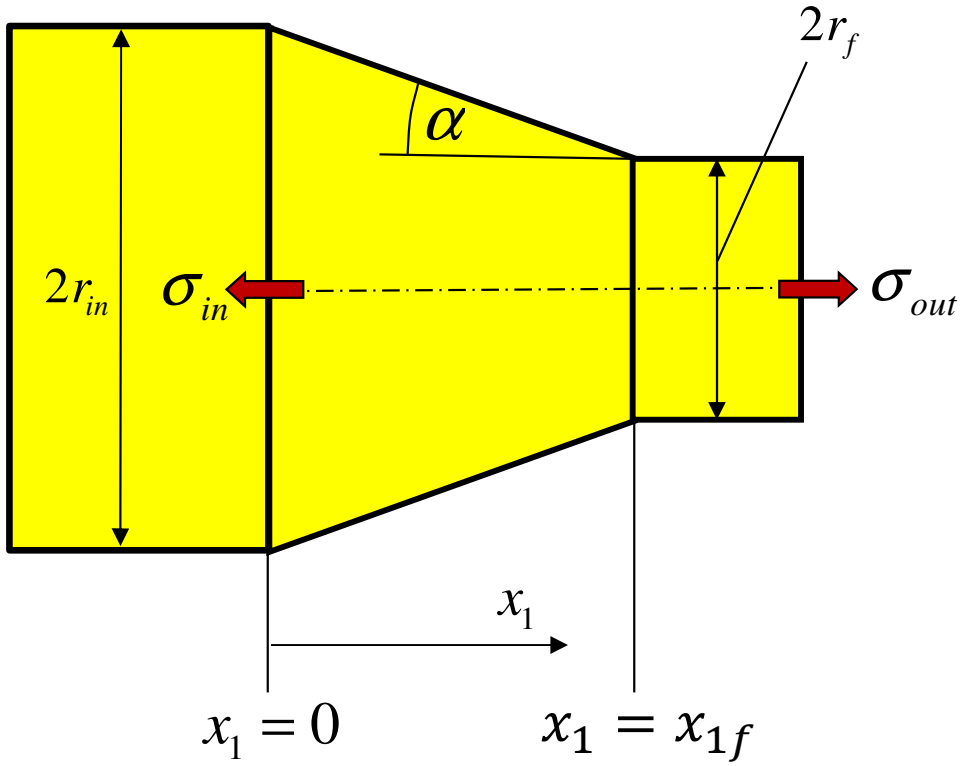
Equilibrium Approach

Supposed conditions

- 1D stress and strain distribution
- Homogeneous fields perpendicular to the direction of change (flow)
- Simplified flow conditions
- Simplified geometry
- Approximate boundary conditions

Axial symmetric case

Material flow in conical dies



Drawing $(\sigma_{out} < \sigma_{f\,out})$

$x_1 = 0, \quad \sigma_{11} = \sigma_{in} = 0$

$x_1 = x_{1f}, \quad \sigma_{11} = \sigma_{\H{o}ut} < \sigma_{fout}$

Reduction $(\sigma_{in} < \sigma_{fin})$

$x_1 = x_{1f}, \quad \sigma_{11} = \sigma_{out} = 0$

$x_1 = 0, \quad \sigma_{11} = \sigma_{in} < \sigma_{fin}$

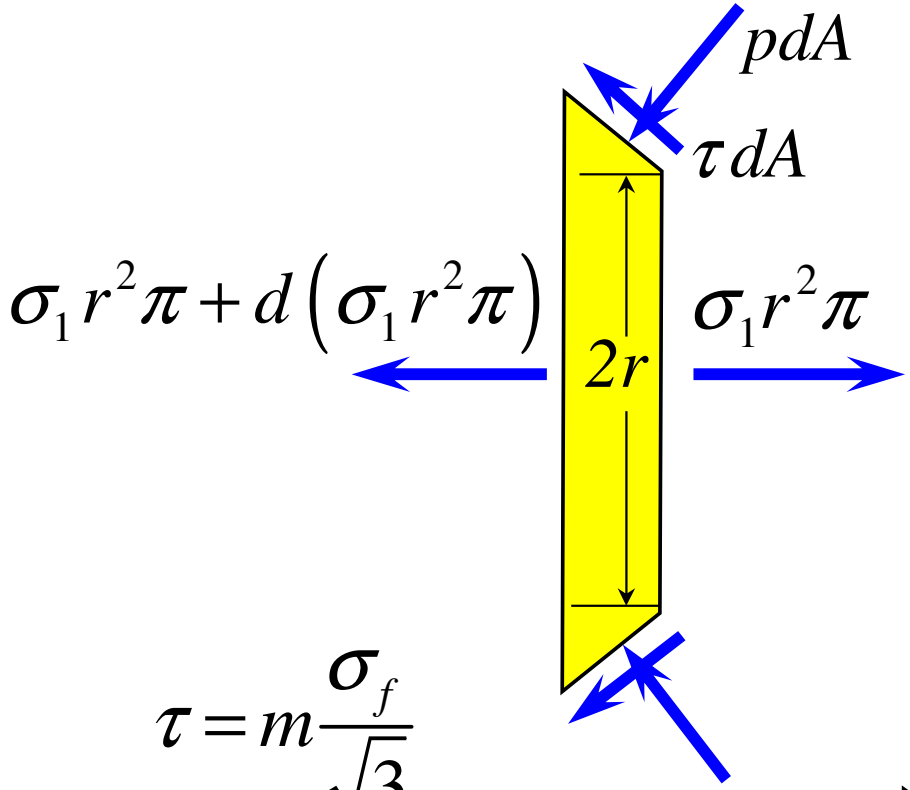
Extrusion

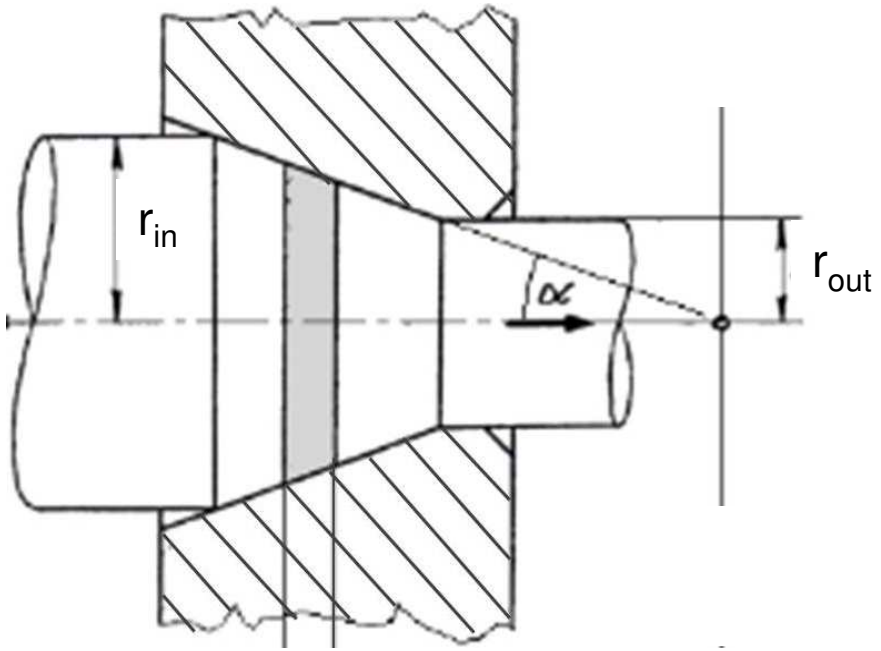
$x_1 = 0, \quad \sigma_{11} = \sigma_{in} + \sigma_{w\,fr} > 0$

$x_1 = x_{1f}, \quad \sigma_{11} = \sigma_{out} = 0$

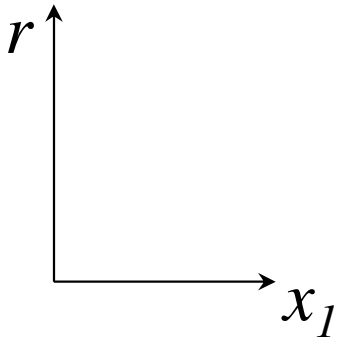
$\sigma_{w\,fr}$ is additional stress from the friction with the recipient wall.)

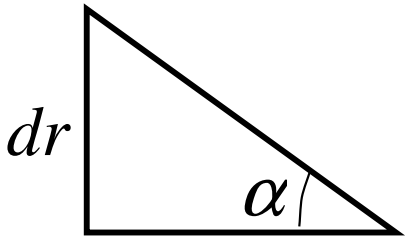
Axial symmetric case



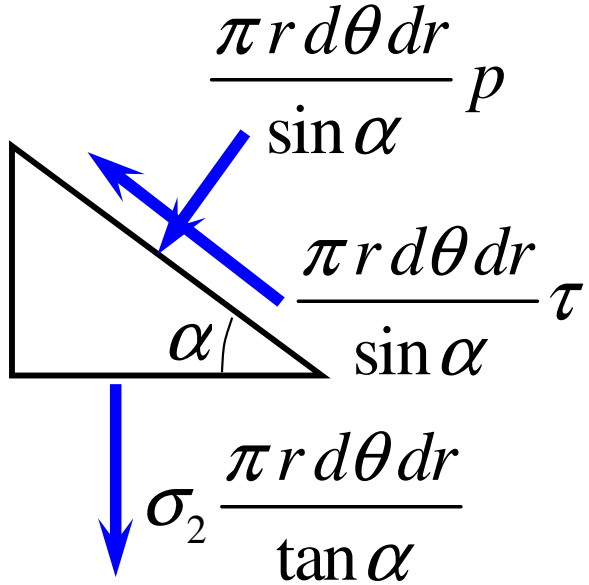


$$\tau = m\frac{\sigma_f}{\sqrt{3}}$$





$$\frac{dr}{\tan\alpha}$$



Axial symmetric case

$$\sum F_i = 0$$

$$\sum F_{x_1} = -d(\sigma_1 r^2 \pi) - 2\,\tau\,\pi\,r\,dr\,\cot\alpha - 2\,p\,\pi\,r = 0$$

$$d(\sigma_1 r^2) + 2(\tau\cot\alpha + p) r\,dr = 0$$

$$d(\sigma_1 r^2) + 2\left(m\frac{\sigma_f}{\sqrt{3}}\cot\alpha + p\right) r\,dr = 0$$

$$\sum F_r = \tau\,r\,dr\,d\theta - p\,r\,dr\,d\theta\cot\alpha - \sigma_r\,r\,dr\,d\theta\cot\alpha$$

$$p = \tau\tan\alpha - \sigma_r = m\frac{\sigma_y}{\sqrt{3}}\tan\alpha - \sigma_r$$

$$\sigma_1 - \sigma_r = \sigma_f \qquad \frac{d\sigma_1}{4\sigma_1 + H\sigma_f} = \frac{dr}{r}$$

$$H = m\sqrt{3}\left(\cot\alpha - \tan\alpha - \sqrt{3}\right) \qquad r = r_{in},\ \ \sigma_1 = 0$$

$$\sigma_1 = \frac{H}{4}\sigma_f\left(\left(\frac{r}{r_{be}}\right)^4 - 1\right) \qquad \sigma_r = \frac{H}{4}\sigma_f\left(\left(\frac{r}{r_{be}}\right)^4 - 1\right) - \sigma_f$$

$$\sigma_f = C_1 + C_2\bar{\varphi}^n \qquad \bar{\varphi} = 2\ln\frac{r_{in}}{r}$$

Axisymmetric case

Coulomb friction

$$\tau = \mu p$$

$$r d\sigma_1 + 2\sigma_1 dr + 2 p dr(1 + \mu \cot\alpha) = 0$$

$$p = \tau \tan\alpha - \sigma_r = p\mu \tan\alpha - \sigma_r \rightarrow p(1 - \mu \tan\alpha) = -\sigma_r$$

$$\sigma_1 - \sigma_r = \sigma_y$$

$$\frac{d\sigma_1}{\sigma_1 B - \sigma_y(1+B)} = \frac{2dr}{r}$$

$$\sigma_1 = \frac{Cr^{2B}}{B} + \frac{1+B}{B}\sigma_y,\ r = r_x,\ \sigma_1 = \sigma_{1x}$$

$$\frac{\sigma_1}{\sigma_y} = \frac{1+B}{B}\left[1 - \left(\frac{r}{r_x}\right)^{2B}\right] + \frac{\sigma_{1x}}{\sigma_y}\left(\frac{r}{r_x}\right)^{2B}$$

$$1 + B = \left(1 + \frac{\mu}{\tan\alpha}\right)\frac{1}{1 - \mu \tan\alpha}$$

Average yield stress

For the solution of differential equations, we assume that the yield stress is constant during the process (by using the average value).

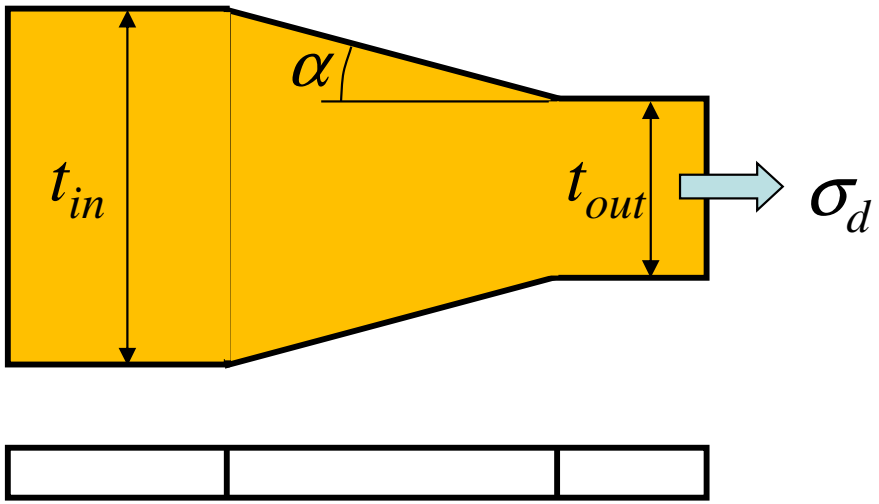
$$\sigma_{f_{average}} = \tilde{\sigma}_f = \frac{1}{\bar{\varphi}_{max} - \bar{\varphi}_{min}} \int_{\bar{\varphi}_{min}}^{\bar{\varphi}_{max}} \sigma_f \, d\bar{\varphi}$$

$$\sigma_f = C_1 + C_2 \bar{\varphi}^n$$

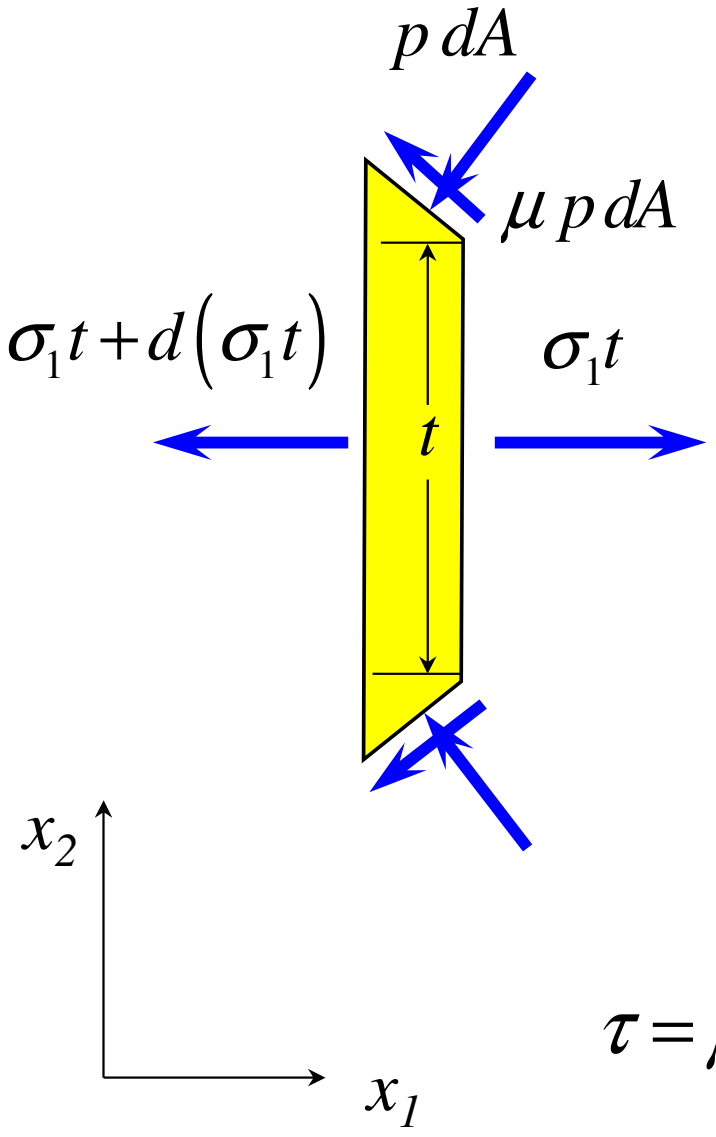
$$\tilde{\sigma}_f = \frac{1}{\bar{\varphi}_{max} - \bar{\varphi}_{min}} \left[C_1 (\bar{\varphi}_{max} - \bar{\varphi}_{min}) + \frac{C_2}{n+1} \left(\bar{\varphi}_{max}^{n+1} - \bar{\varphi}_{min}^{n+1} \right) \right]$$

Flow trough a narrowing die

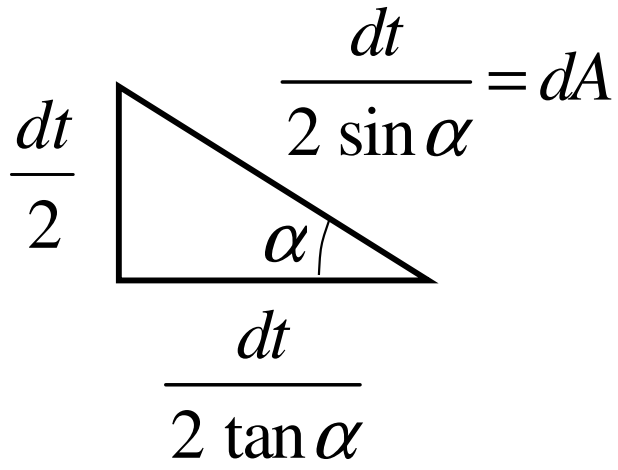
Plain strain case

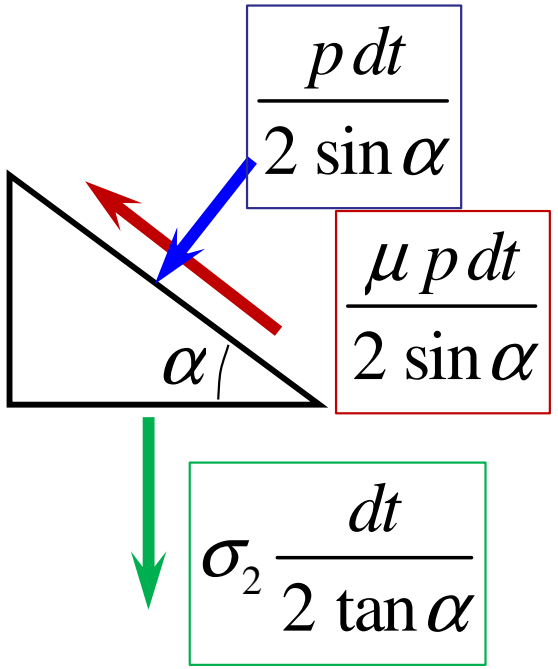


Plain strain case



$$\tau = \mu\, p$$





Plain strain case

$$\sum F_i = 0$$

$$\sum F_{x_2} = 0 = \mu p \frac{dt}{2} - \frac{p dt}{2\tan\alpha} - \sigma_2 \frac{dt}{2\tan\alpha} = 0$$

$$\sigma_2 = p(\mu\tan\alpha - 1) \approx -p$$

$$\sum F_{x_1} = 0 = \sigma_1 t - \sigma_1 t - d(\sigma_1 t) - 2\frac{\mu p dt}{2\tan\alpha} - 2\frac{p}{2} dt = 0$$

$$\sigma_1 dt + t d\sigma_1 = -p dt \cdot \left(\frac{\mu}{\tan\alpha} + 1\right) =$$

$$= -\sigma_2 dt \left(\frac{\mu}{\tan\alpha} + 1\right)(1 - \mu\tan\alpha) - \sigma_1 dt = \sigma_2 (B+1) dt$$

$$d\sigma_1 = (\sigma_2 (1+B) - \sigma_1)\frac{dt}{t}$$

$$B = \left(\frac{\mu}{\tan\alpha} + 1\right)(1 - \mu\tan\alpha) - 1$$

Plain strain case

Plain strain $\xi_2 = 0, \quad \xi_{ij} = \dot{\lambda}\sigma'_{ij} \rightarrow \quad 0 = \dot{\lambda}\left(2\sigma_2 - \sigma_1 - \sigma_3\right)$

$$\sigma_2 = \frac{\sigma_1 + \sigma_3}{2},$$

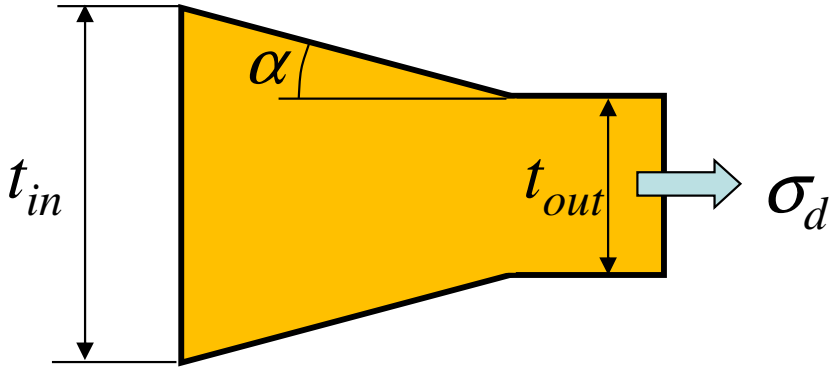
$$yield\ criteria: \quad \bar{\sigma} = \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_1 - \sigma_3\right)^2 + \left(\sigma_2 - \sigma_3\right)^2} = \sigma_f$$

$$\bar{\sigma} = \frac{\sqrt{3}}{2}\left|\sigma_1 - \sigma_3\right| = \frac{\sqrt{3}}{2}\left(\sigma_1 - \sigma_3\right),$$

$$\sigma_1 > \sigma_3, \quad \sigma_2 = \sigma_1 - \frac{2}{\sqrt{3}}\sigma_f$$

$$d\sigma_1 = \left[\left(1+B\right)\left(\sigma_1 - \frac{2}{\sqrt{3}}\sigma_y\right) - \sigma_1\right]\frac{dt}{t} \rightarrow \frac{d\sigma_1}{B\sigma_1 - \frac{2}{\sqrt{3}}\sigma_y\left(1+B\right)} = \frac{dt}{t}$$

Plain strain case



$$\frac{d\sigma_1}{B\sigma_1 - \frac{2}{\sqrt{3}}\sigma_f(1+B)} = \frac{dt}{t}$$

Boundary conditions: drawing

$$t = t_{in},\quad \sigma_1 = 0$$

$$t = t_{out},\quad \sigma_1 = \sigma_d$$

$$\int_{\sigma_1=0}^{\sigma_1=\sigma_d} \frac{d\sigma_1}{B\sigma_1 - \frac{2}{\sqrt{3}}\sigma_f(1+B)} = \int_{t=t_{in}}^{t=t_{out}} \frac{dt}{t} \rightarrow \sigma_d = \frac{2}{\sqrt{3}}\sigma_f \frac{(1+B)}{B}\left[1-\left(\frac{t_{out}}{t_{in}}\right)^B\right]$$

Plain strain case

$$\sigma_1(t)=\frac{2}{\sqrt{3}}\sigma_f\frac{(1+B)}{B}\left[1-\left(\frac{t}{t_{in}}\right)^B\right],$$

$$\sigma_2(t)=\frac{2}{\sqrt{3}}\sigma_f\frac{(1+B)}{B}\left[1-\left(\frac{t}{t_{in}}\right)^B\right]-\frac{2}{\sqrt{3}}\sigma_f$$

$$\sigma_f=\text{const.}$$

$$p=\frac{\sigma_2}{\mu\tan\alpha-1}$$

$$\frac{d\sigma_1}{dt}-\left[(1+B)\left(\sigma_1(t)-\frac{2}{\sqrt{3}}\sigma_f(t)\right)-\sigma_1(t)\right]\frac{1}{t}=0,\qquad \bar{\varphi}=\frac{2}{\sqrt{3}}\ln\frac{t_{in}}{t}$$

Steps of equilibrium approach

- Choosing a direction characteristic of the deformation/stress.
- Defining the equilibrium of the forces acting on an elemental slice of the body, which is perpendicular to the chosen direction
- Construction of a differential equation (1D)
- Flow condition, reduction of the number of variables
- Solution of the differential equation with the boundary conditions

Energy Approach

Total power of outer forces

$$J = \int_V \sigma_f \bar{\xi} dV + \int_{A_\Gamma} \tau |\Delta v| \, dA_\Gamma - \int_{A_t} T_i \, v_i$$

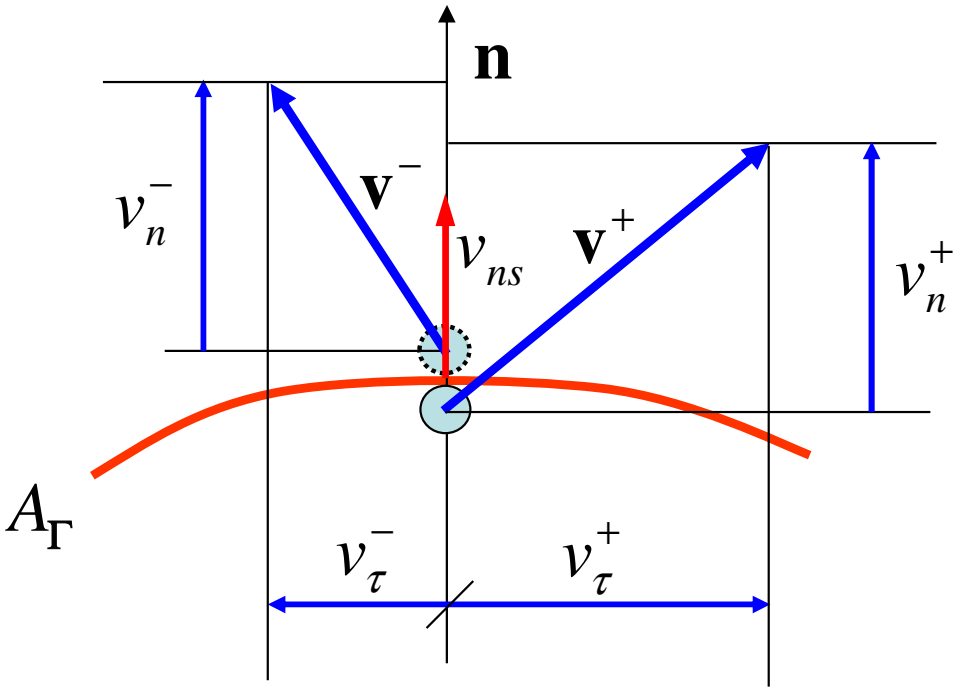
From the numerous kinematically possible velocity fields, the expression above is minimal for the actual (real) one.

First part: the power of inner forces
Second part: the power of discontinuity surfaces
Third part: the power of external constraints (e.g. wire drawing)

$$J = \dot{W}_i + \dot{W}_\Gamma - \dot{W}_t$$

Kinematic boundary conditions and incompressibility must hold for the real velocity field.

Discontinuity on boundaries

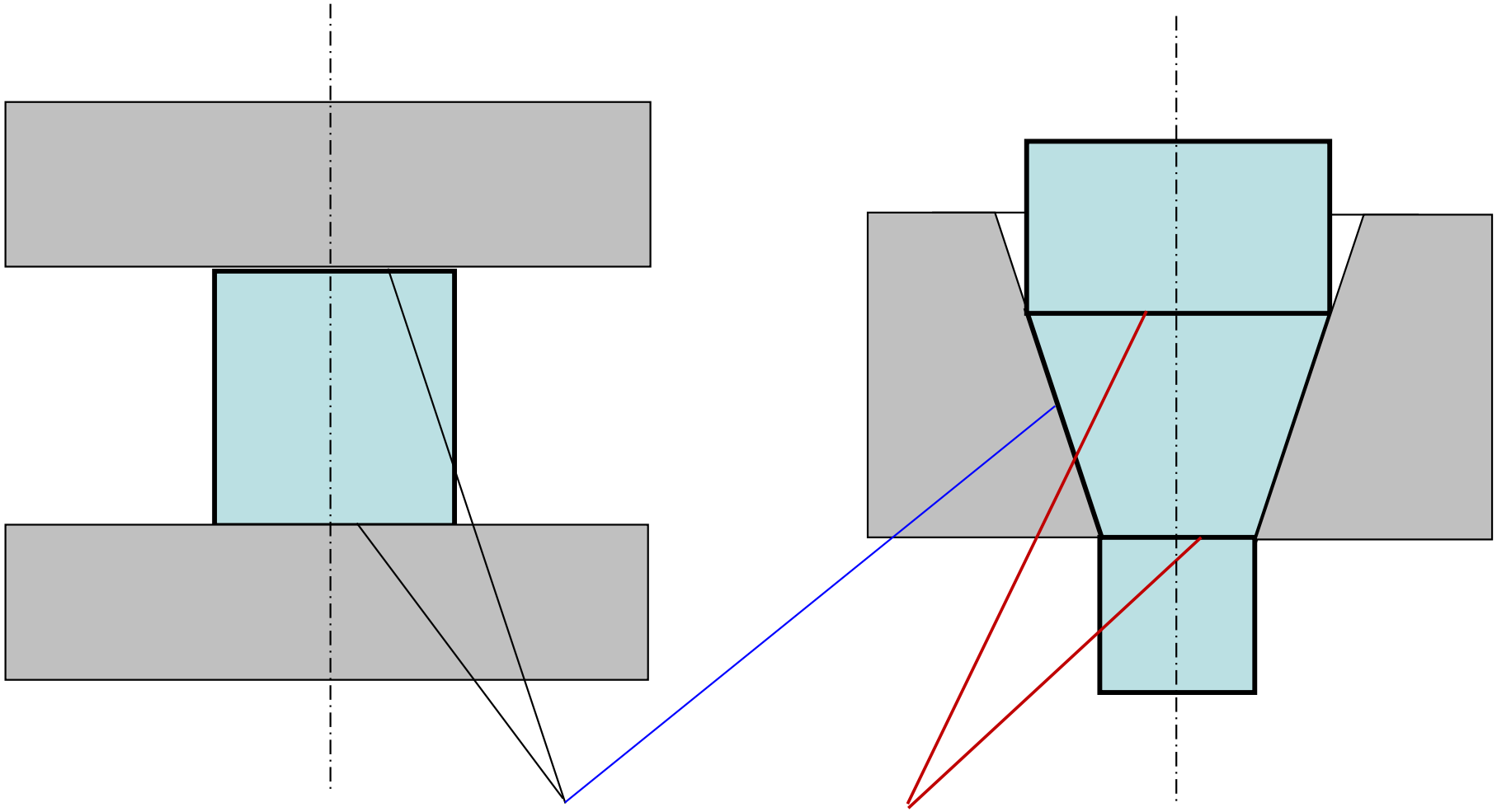


$$v_n^+ = v_n^-$$

$$\Delta v = v_\tau^+ - v_\tau^- \neq \mathbf{0}$$

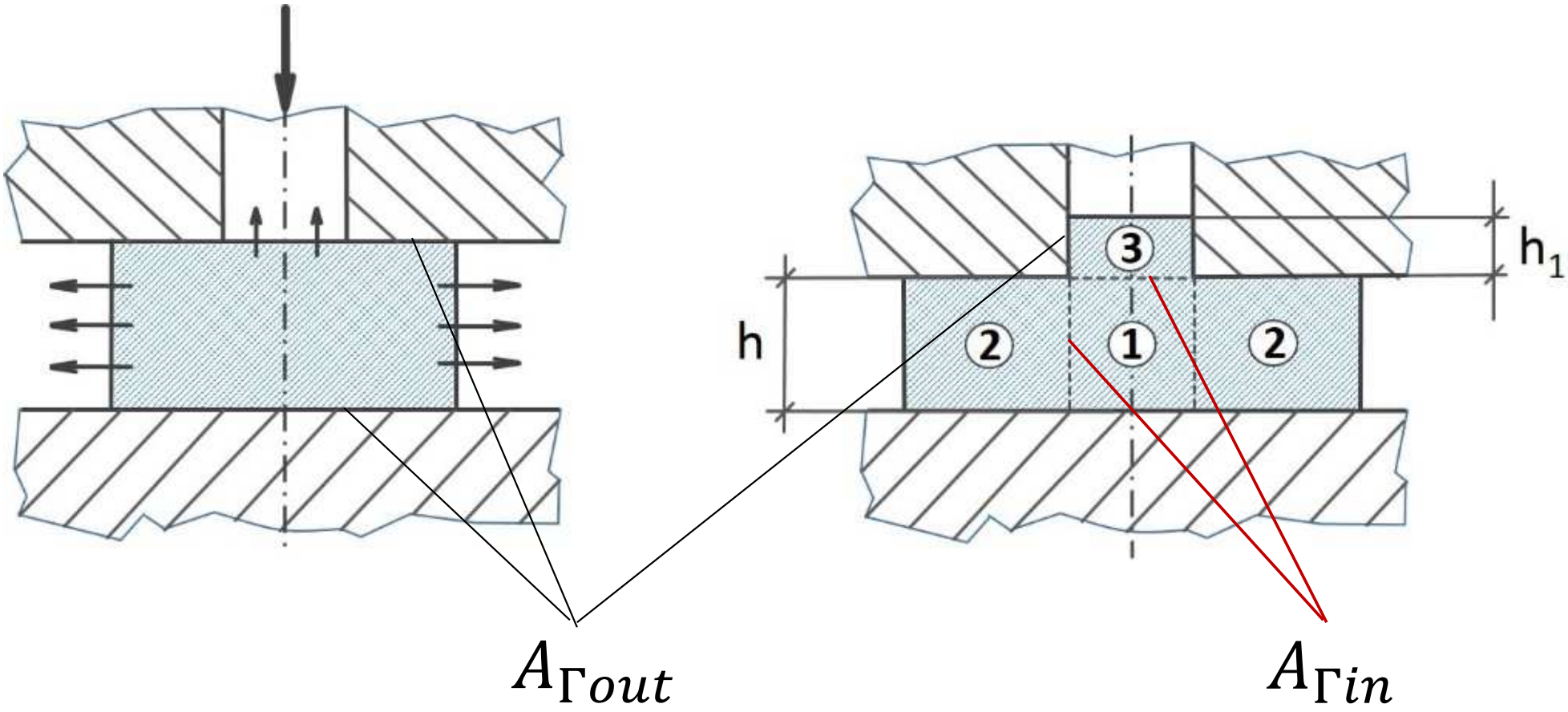
$$\Delta\bar{\varepsilon} = \frac{|\Delta v|}{\sqrt{3}(v_n - v_{ns})}; \qquad \bar{\sigma}_\Gamma = \frac{1}{\Delta\bar{\varepsilon}} \int_{\bar{\varepsilon}}^{\bar{\varepsilon}+\Delta\bar{\varepsilon}} \sigma_f d\bar{\varepsilon}$$

Discontinuity on boundaries



Discontinuity on boundaries

An application – the final shape is uncertain



$$J = J_1 + J_2 + J_3 = F(h_1) \Rightarrow \frac{\partial J}{\partial h_1} = 0 \Rightarrow h_1$$

Steps of energy approach

- Based on the actual flow, define all components of the velocity field.
- Defining the other components based on incompressibility (solution of a differential equation)
- Construction of kinematically possible strain rate fields
- Power of inner forces and discontinuity surfaces
- Finding the extrema of the functional.

Thank you for your attention !